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Discrete Mathematics 303 (2005) 235–243

DISCRETE MATHEMATICS

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Triangulations on closed surfaces which quadrangulate other surfaces II

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Received 19 December 2002; received in revised form 3 September 2003; accepted 8 December 2004

Abstract

It has already been proved that given two closed surfaces F_1^2 and F_2^2 with $2\chi(F_1^2) - \chi(F_2^2) \geqslant 4$, there exists a triangulation on F_1^2 which can be embedded on F_2^2 as a quadrangulation. In this paper we refine that result, showing that there exists an integer g_0 such that for any two closed surfaces with genus $g_1 \geqslant g_0$ and genus g_2 satisfying $2\chi(F_1^2) - \chi(F_2^2) \geqslant O(g_1)$, there exists a triangulation of the first surface which can be re-embedded on the second as a quadrangulation. Moreover, on the right-hand side of the inequality, we obtain a concrete expression which is asymptotically $O(g_1)$. We also obtain similar results for non-orientable surfaces.

Keywords: Triangulation; Quadrangulation; Complete graph

1. Introduction

An *n-gonal embedding* of a simple graph G on a closed surface F^2 is an embedding of G on F^2 such that each face is bounded by a cycle of length n. In particular, a 3-gonal embedding of G on F^2 is said to be *triangular* and its image on F^2 is called a *triangulation* on F^2, while a 4-gonal embedding is *quadrangular* and its image is a *quadrangulation*. Following the convention in topological graph theory, we do not regard K_3 on the sphere as a triangulation. Thus, the smallest triangulation is the tetrahedron, which is K_4 on the sphere, and it can be embedded on the projective plane as a quadrangulation. Also, the octahedron,

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2004.12.025

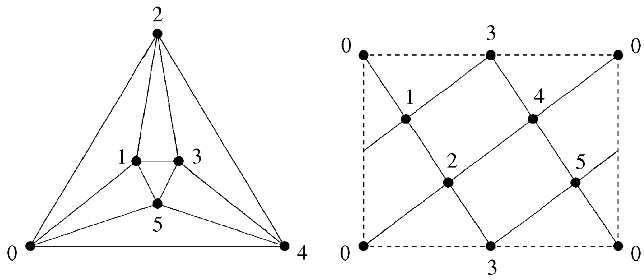

Fig. 1. Octahedron on the torus.

which is a typical triangulation on the sphere, admits a quadrangular embedding on the torus as shown in Fig. 1.

Recently, Nakamoto et al. [3] have shown that for any given closed surface except the sphere, there is a triangulation on the sphere which can be embedded on the closed surface as a quadrangulation. The author has extended their arguments to increase the genus of the surfaces where we embed graphs as triangulations. If there exists such a triangulation G on F_1^2, the number of its vertices and edges must be restricted by Euler's formula as follows:

$$|V(G)| = 3\chi(F_1^2) - 2\chi(F_2^2), \quad |E(G)| = 6\chi(F_1^2) - 6\chi(F_2^2),$$

where we denote the Euler characteristic of a closed surface F^2 by $\chi(F^2)$. The author has already proved the following theorem:

Theorem 1 (*Suzuki [7]*). *Given two closed surfaces F_1^2 and F_2^2 with $2\chi(F_1^2) - \chi(F_2^2) \geqslant 4$, there exists a triangulation on F_1^2 which can be embedded on F_2^2 as a quadrangulation.*

In a few words, the above theorem states that if there is a certain amount of difference between the genera of closed surfaces F_1^2 and F_2^2, there exists such a triangulation on F_1^2. We shall reduce the difference and establish the following theorem in this paper.

Theorem 2. *There exists a natural number g_0 such that for any two closed surfaces F_1^2 of genus $g_1 \geqslant g_0$ and F_2^2 satisfying $2\chi(F_1^2) - \chi(F_2^2) \geqslant O(g_1)$, there exists a triangulation on F_1^2 which can be embedded on F_2^2 as a quadrangulation.*

At first, we shall introduce a notion called "slit–flip sum" in Section 2. Slit–flip sum is a method of pasting two graphs to construct a new graph. This method has already been introduced in [3], but we shall refine it for our purpose. In Section 2, first, we shall discuss the existence of complete graphs which triangulate closed surfaces and quadrangulate other surfaces. Using those complete graphs, we shall construct the required graphs in Theorems 4–7.

2. Slit–flip sums

Let G_i be a graph which can be embedded on closed surfaces Σ_i^2 as a triangulation T_i and on F_i^2 as a quadrangulation Q_i for $i = 1, 2$. If there is a path $u_i v_i w_i$ in G_i which forms a corner of a face in T_i and also does in Q_i, we call it a *useful corner* for G_i at v_i. If we can find such useful corners $u_1 v_1 w_1$ and $u_2 v_2 w_2$ in the embeddings of G_1 and G_2, respectively, cut open the two closed surfaces Σ_1^2 and Σ_2^2, each including T_1 and T_2, along the edges $u_1 v_1$ and $u_2 v_2$, respectively, and paste them along the resulting boundaries so that u_1 is identified with v_2 and v_1 with u_2, and so that the union of two faces $u_1 v_1 w_1$ and $u_2 v_2 w_2$ forms a rectangular region with a diagonal $u_1 v_1 = v_2 u_2$. Replace this diagonal with a single edge $w_1 w_2$ to eliminate the multiple edges between u_1 and v_1, and v_2 and u_2. Let $T_1 \ddagger T_2$ denote the resulting triangulation on the closed surface Σ^2 obtained from the two slitted surfaces. In a similar way, we can construct a new quadrangulation $Q_1 \ddagger Q_2$ on a closed surface F^2 obtained from F_1^2 and F_2^2 slitted along $u_1 v_1$ and $u_2 v_2$.

It is clear that $T_1 \ddagger T_2$ and $Q_1 \ddagger Q_2$ are isomorphic to a common graph, say $G_1 \ddagger G_2$. We say that $G_1 \ddagger G_2$ is obtained from G_1 and G_2 by a *slit–flip sum*. The following lemma is an important result for useful corner and enables us to carry out slit–flip sum repeatedly.

Lemma 3. *Any graph $G_1 \ddagger G_2$ obtained by a slit–flip sum has two useful corners.*

Proof. Let $u_i v_i w_i$ be a useful corner in a graph G_i, and suppose that a slit–flip sum flips $u_1 v_1 = v_2 u_2$ to $w_1 w_2$ to obtain $G_1 \ddagger G_2$. Then both $u_1 w_2 w_1$ and $v_1 w_1 w_2$ form two corners in $T_1 \ddagger T_2$ and also in $Q_1 \ddagger Q_2$. Thus, they are useful corners in $G_1 \ddagger G_2$. □

Now we shall consider performing slit–flip sums at "two places." If both G_1 and G_2 have two disjoint useful corners, $u_i v_i w_i$ and $u_i' v_i' w_i'$, then we can perform the slit–flip sums at these corners at the same time. Denote the resulting graph, triangulation and quadrangulation by $G_1 \ddagger\ddagger G_2$, $T_1 \ddagger\ddagger T_2$ and $Q_1 \ddagger\ddagger Q_2$, respectively. Then $T_1 \ddagger\ddagger T_2$ triangulates a closed surface homeomorphic to $\Sigma_1^2 \# \Sigma_2^2$ with one handle attached while $Q_1 \ddagger\ddagger Q_2$ quadrangulates a closed surface homeomorphic to $F_1^2 \# F_2^2$ with one handle attached. It should be noticed that such surfaces need not be orientable even if all Σ_i^2 and F_i^2 are orientable. Let $\Sigma_1^2 \ddagger\ddagger \Sigma_2^2$ and $F_1^2 \ddagger\ddagger F_2^2$ denote these surfaces. To control their orientability, we define the "type" of a pair of useful corners as follows.

Let G be a graph which can be embedded on Σ^2 as a triangulation and on F^2 as a quadrangulation and suppose that both surfaces Σ^2 and F^2 are orientable. Let uvw and $u'v'w'$ be two disjoint useful corners for G. They are said to be *coherent* on Σ^2 (or F^2) if they induce the same orientation over Σ^2 (or F^2) and to be *incoherent* otherwise. Such a pair of useful corners is said to be of *type* $(\pm, \pm)$, where "+" (or "−") means that they are coherent (or incoherent) on Σ^2 and F^2 in order. It is easy to see that the octahedron has four pairs of disjoint useful corners of all types. For example, the pair of corners $\{051, 423\}$ in Fig. 1 are of type $(+, -)$.

Let G_i, T_i, Q_i, Σ_i^2 and F_i^2 be as above. It is clear that $\Sigma_1^2 \ddagger\ddagger \Sigma_2^2$ is orientable, if and only if both Σ_1^2 and Σ_2^2 are orientable, and the pair of useful corners for G_1 has the same coherency as that for G_2. In particular, when G_2 is the octahedron, we say that $G_1 \ddagger\ddagger G_2$ is obtained

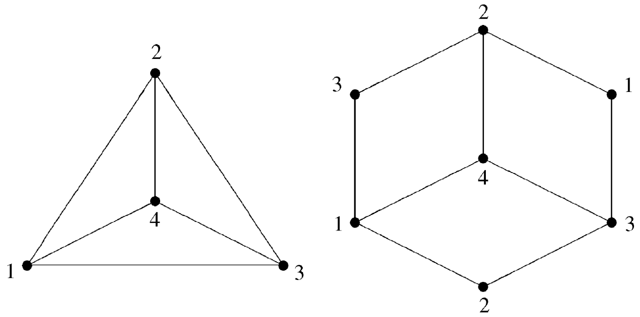

Fig. 2. Tetrahedron on the projective plane.

from G_1 by *adding a slit–flip handle*. It is easy to see that a slit–flip handle decreases $\chi(\Sigma_1^2)$ by 2 and $\chi(F_1^2)$ by 4. If both Σ_1^2 and F_1^2 are orientable, then we can control the orientability of $\Sigma_1^2 \ddagger\ddagger \Sigma_2^2$ and of $F_1^2 \ddagger\ddagger F_2^2$, performing the slit–flip handle addition at a suitable pair of useful corners.

Fig. 2 presents two embeddings of the complete graph K_4. The picture on the left-hand side shows a triangulation on the sphere, where identifying the antipodal boundary points on the hexagon shown on the right-hand side yields a quadrangulation on the projective plane. It is easy to check that all corners in the tetrahedron are useful corners. A slit–flip sum with this tetrahedron adds one crosscap to F_1^2, not changing the homeomorphism type of Σ_1^2. We call this slit–flip sum simply *adding a crosscap*. On the other hand, if a slit–flip sum with the octahedron in Fig. 1 adds one orientable handle to F_1^2, not changing the homeomorphism type of Σ_1^2, then we call this operation *adding a handle*.

3. Proof of Theorem 2

To prove Theorem 2, we split it into the following four theorems accordingly to the orientabilities of the surfaces. We often use "k_i" instead of "g_i" to distinguish the genus of a non-orientable closed surface from that of an orientable one.

Theorem 4. *There exists a natural number g_0 such that given two orientable closed surfaces F_1^2 of genus $g_1 \geqslant g_0$ and F_2^2 of genus g_2 with*

$$2g_2 - 3g_1 - \left\lceil \frac{-331 + 19\sqrt{1+48g_1}}{12} \right\rceil \geqslant -1,$$

there exists a triangulation on F_1^2 which can be embedded on F_2^2 as a quadrangulation.

Theorem 5. *There exists a natural number g_0 such that given an orientable closed surface F_1^2 of genus $g_1 \geqslant g_0$ and a non-orientable closed surface F_2^2 of genus k_2 with*

$$k_2 - 3g_1 - \left\lceil \frac{-91 + 11\sqrt{1+48g_1}}{12} \right\rceil \geqslant -1,$$

there exists a triangulation on F_1^2 which can be embedded on F_2^2 as a quadrangulation.

Theorem 6. *There exists a natural number k_0 such that given a non-orientable closed surface F_1^2 of genus $k_1 \geqslant k_0$ and an orientable closed surface F_2^2 of genus g_2 with*

$$2g_2 - k_1 - \left\lceil \frac{-331 + 19\sqrt{1 + 24k_1}}{12} \right\rceil \geqslant -1,$$

there exists a triangulation on F_1^2 which can be embedded on F_2^2 as a quadrangulation.

Theorem 7. *There exists a natural number k_0 such that given two non-orientable closed surfaces F_1^2 of genus $k_1 \geqslant k_0$ and F_2^2 of genus k_2 with*

$$2k_2 - 3k_1 - \left\lceil \frac{-91 + 11\sqrt{1 + 24k_1}}{6} \right\rceil \geqslant -2,$$

there exists a triangulation on F_1^2 which can be embedded on F_2^2 as a quadrangulation.

To simplify the description below, we shall use a slightly informal notation $G_{\varepsilon_1\varepsilon_2}(g_1, g_2)$ with closed surfaces F_1^2 and F_2^2, where $\varepsilon_i \in \{+, -\}$ stands for the orientability of F_i^2. For example, $G_{+-}(g_1, k_2)$ corresponds to the phrase "a certain graph which can be embedded on the orientable $(+)$ closed surface F_1^2 of genus g_1 as a triangulation and on the non-orientable $(-)$ closed surface F_2^2 of genus k_2 as a quadrangulation". We prepare the following lemmas which are useful to prove Theorems 4–7.

Lemma 8. *If there exists $G_{++}(g_1, g_2)$ having two disjoint useful corners, then there exists $G_{++}(g'_1, g'_2)$ with $g'_1 \geqslant g_1$ and $g'_2 \geqslant 2g'_1 + g_2 - 2g_1$.*

Proof. We use $G_{++}(g_1, g_2)$ as the base of adding handles. It has already been explained in the previous section that adding a handle increases the genus of F_2^2 by 1, not changing the homeomorphism type of F_1^2. Adding a slit–flip handle to $G_{++}(g_1, g_2)$ increases the genus of F_1^2 by 1 and increases the genus of F_2^2 by 2.It is easy to see that we can construct our desired graph $G_{++}(g'_1, g'_2)$ with $g'_1 \geqslant g_1$, $g'_2 \geqslant 2g'_1 + g_2 - 2g_1$ by adding handles and adding slit–flip handles repeatedly. □

Lemma 9. *If there exists $G_{+-}(g_1, k_2)$ having two disjoint useful corners, then there exists $G_{+-}(g'_1, k'_2)$ with $g'_1 \geqslant g_1$ and $k'_2 \geqslant 4g'_1 + k_2 - 4g_1$.*

Proof. Use $G_{+-}(g_1, k_2)$ as the base of adding handles. Adding a crosscap increases the genus of F_2^2 by 1, not changing the homeomorphism type of F_1^2. Adding a slit–flip handle to the $G_{+-}(g_1, k_2)$ increases the genus of F_1^2 by 1 and increases the genus of F_2^2 by 4. So we can construct $G_{+-}(g'_1, k'_2)$ satisfying the conditions of the lemma. □

In the following two cases, we have to notice the parity of the genera of non-orientable closed surfaces where triangulations are embedded, since, adding a slit–flip handle increases the genus of each of these surfaces by two.

Lemma 10. *If there exists $G_{-+}(k_1, g_2)$ having two disjoint useful corners with k_1 even (or odd), then there exists $G_{-+}(k'_1, g'_2)$ such that k'_1 is even (or odd), $k'_1 \geqslant k_1$ and $g'_2 \geqslant k'_1 + g_2 - k_1$.*

Proof. The same proof as that of Lemma 8 works except the fact that adding a slit–flip handle to $G_{-+}(k_1, g_2)$ increases the genus of F_1^2 by 2 and increases the genus of F_2^2 by 2. Adding handle can control the parity of the latter freely. □

Lemma 11. *If there exists $G_{--}(k_1, k_2)$ having two disjoint useful corners with k_1 even (or odd), then there exists $G_{--}(k'_1, k'_2)$ such that k'_1 is even (or odd), $k'_1 \geqslant k_1$ and $k'_2 \geqslant 2k'_1 + k_2 - 2k_1$.*

Proof. Use adding crosscaps and adding slit–flip handles. Adding a crosscap increases the genus of F_2^2 by 1, not changing the homeomorphism type of F_1^2 in the same way as in the proof of Lemma 9. On the other hand, adding a slit–flip handle to $G_{--}(k_1, k_2)$ increases the genus of F_1^2 by 2 and increases the genus of F_2^2 by 4. □

We would like to use the complete graph K_n with n vertices as $G_{\varepsilon_1\varepsilon_2}(g_1, g_2)$ in Lemmas 8–11. Here, the existence of two disjoint useful corners is guaranteed. Because we can find many disjoint useful corners in two way embedded K_n by thinking symmetry of complete graph's labeling.

Ringel [5] has shown that K_n can be embedded on a closed surface as a triangulation if n satisfies a suitable condition, as follows. For quadrangular embeddings of K_n on non-orientable closed surfaces, Hartsfield and Ringel [1] have shown a similar result. We rewrite their result in the same way as for triangulations. Quadrangular embeddings of K_n on orientable closed surfaces have been studied by Hartsfield, but her results have not been published. Recently, Lawrencenko and Yang [2] have carried out and completed their work, as follows below in Theorem 14.

Theorem 12 (*Ringel [5]*). *The complete graph K_n with n vertices can triangulate an orientable (or non-orientable) closed surface if and only if $n \equiv 0, 3, 4$ or $7 \bmod 12$ ($n \equiv 0, 1, 3$ or $4 \bmod 6$ and $n \neq 7$).*

Theorem 13 (*Hartsfield and Ringel [1]*). *The complete graph K_n with n vertices can quadrangulate a non-orientable closed surface if and only if $n \equiv 0$ or $1 \bmod 4$.*

Theorem 14 (*Lawrencenko and Yang [2]*). *The complete graph K_n with n vertices can quadrangulate an orientable closed surface if and only if $n \equiv 0$ or $5 \bmod 8$.*

Proof of Theorem 4. We shall construct triangulations on orientable closed surfaces each of which quadrangulates another orientable closed surface. From Theorems 12 and 14 it

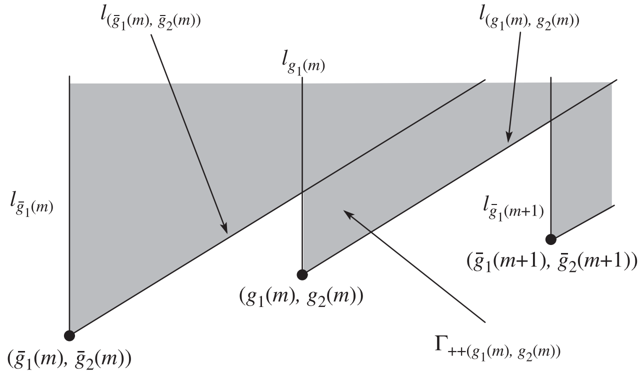

Fig. 3. Γ_{++}.

follows that the complete graph K_n with n vertices can triangulate and quadrangulate two different orientable closed surfaces if and only if $n \equiv 0$ or $16 \bmod 24$.So, we use K_{24m} $(m \geqslant 1)$ and K_{24m-8} $(m \geqslant 1)$ as $G_{++}(g_1, g_2)$ in Lemma 8.

By calculating using Euler's formula, we can easily conclude that K_{24m} can be embedded on an orientable closed surface having genus $g_1(m) = 48m^2 - 14m + 1$ as a triangulation and embedded on another orientable closed surface having genus $g_2(m) = 72m^2 - 15m + 1$ as a quadrangulation. Similarly, K_{24m-8} can be embedded on two different orientable closed surfaces having genus $\bar{g}_1(m) = 48m^2 - 46m + 11$ and $\bar{g}_2(m) = 72m^2 - 63m + 14$.

Put $\Gamma_{++}(g_1, g_2) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \geqslant g_1,\ y \geqslant 2x + g_2 - 2g_1\}$, corresponding to the set of integer points (g_1', g_2') in Lemma 8. We denote the two lines $x = g_1$ and $y = 2x + g_2 - 2g_1$ by l_{g_1} and $l_{(g_1,g_2)}$, respectively. In other words, $\Gamma_{++}(g_1, g_2)$ is the set of integer points lying in the wedge region between l_{g_1} and $l_{(g_1,g_2)}$, (see Fig. 3). By Lemma 8, we can construct the required graphs $G_{++}(g_1', g_2')$ for

$$(g_1', g_2') \in \Gamma_{++}^{\infty} = \bigcup_{m=1}^{\infty} \Gamma_{++}(g_1(m), g_2(m)) \cup \bigcup_{m=1}^{\infty} \Gamma_{++}(\bar{g}_1(m), \bar{g}_2(m)).$$

Let $h_1(x) = \left\lceil \frac{-331+19\sqrt{1+48x}}{12} \right\rceil$ and $h_2(x) = \left\lceil \frac{-91+11\sqrt{1+48x}}{12} \right\rceil$. Then the curve $y = h_1(x) + \frac{3x}{2} - \frac{1}{2}$ passes through all of the intersection points of $l_{g_1(m)}$ and $l_{(\bar{g}_1(m),\bar{g}_2(m))}$, while $y = h_2(x) + \frac{3x}{2} - \frac{1}{2}$ passes through those of $l_{\bar{g}_1(m+1)}$ and $l_{(g_1(m),g_2(m))}$ for $m \geqslant 1$. It is easy to see that there exists g_0 such that if $x \geqslant g_0$, then $h_1(x) \geqslant h_2(x)$. Thus, we can easily conclude that $\Lambda_{++} = \left\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \geqslant g_0,\ y \geqslant h_1(x) + \frac{3x}{2} - \frac{1}{2}\right\} \subset \Gamma_{++}^{\infty}$. Therefore, there exists $G_{++}(g_1', g_2')$ for each$(g_1', g_2') \in \Lambda_{++}$, which is nothing but the condition in the theorem. $\square$

We shall prove the other three theorems in the same way as in the proof of Theorem 4. We use the similar notations $\Gamma_{+-}(g_1, k_2)$, $\Gamma_{-+}(k_1, g_2)$ and $\Gamma_{--}(k_1, k_2)$ in the proofs of Theorem 5–7 related to Lemma 9–11.

Proof of Theorem 5. We shall construct triangulations on orientable closed surfaces each of which quadrangulates a non-orientable closed surface. From Theorems 12 and 13 we conclude that the complete graph K_n with n vertices can triangulate an orientable closed surface and quadrangulate a non-orientable closed surface if and only if $n \equiv 0$ or $4 \bmod 12$. It is easy to see that K_{12m} can be embedded on an orientable closed surface having genus $g_1(m) = 12m^2 - 7m + 1$ as a triangulation and embedded on a non-orientable closed surface having genus $k_2(m) = 36m^2 - 15m + 2$ as a quadrangulation. Similarly, we find that K_{12m-8} can be embedded on two different closed surfaces, orientable and non-orientable, having genus $\bar{g}_1(m) = 12m^2 - 23m + 11$ and $\bar{k}_2(m) = 36m^2 - 63m + 28$. By Lemma 9, we can construct the required graphs $G_{+-}(g_1', k_2')$ for

$$(g_1', k_2') \in \bigcup_{m=1}^{\infty} \Gamma_{+-}(g_1(m), k_2(m)) \cup \bigcup_{m=1}^{\infty} \Gamma_{+-}(\bar{g}_1(m), \bar{k}_2(m)).$$

Let

$$h_3(x) = \left\lceil \frac{-19 + 7\sqrt{1 + 48g_1}}{12} \right\rceil.$$

Consider the two curves $y = 2h_3(x) + 3x - 1$ and $y = 2h_2(x) + 3x - 1$ in the same context as in the proof of Theorem 4. □

In the remaining two cases, we have to pay attention to the parity of the genus of the non-orientable closed surface with a triangulation embedded.

Proof of Theorem 6. We shall construct triangulations on non-orientable closed surfaces each of which quadrangulates an orientable closed surface. From Theorems 12 and 14 we can conclude that the complete graph K_n with n vertices can triangulates a non-orientable closed surface and quadrangulates an orientable closed surface if and only if $n \equiv 0$, 13, 16 or $21 \bmod 24$.

First, we construct such graphs which can be embedded on non-orientable closed surfaces having "even" genus as triangulations by the even case of Lemma 10. We prepare K_{24m} $(m \geqslant 1)$ and K_{24m-8} $(m \geqslant 1)$. Clearly, when these complete graphs triangulate non-orientable closed surfaces, the surfaces have even genus. In the same way as in the previous proofs, consider the two curves $y = h_2\left(\frac{1}{2}x\right) + \frac{1}{2}x - \frac{1}{2}$ and $y = h_1\left(\frac{1}{2}x\right) + \frac{1}{2}x - \frac{1}{2}$.

Secondly, we construct such triangulation on non-orientable closed surface having odd genus by the "odd" case of Lemma 10. We prepare K_{24m-11} $(m \geqslant 1)$ and K_{24m-3} $(m \geqslant 1)$. Consider the same two curves $y = h_2\left(\frac{1}{2}x\right) + \frac{1}{2}x - \frac{1}{2}$ and $y = h_1\left(\frac{1}{2}x\right) + \frac{1}{2}x - \frac{1}{2}$ as in the even case, too. These curves lead to the condition in the theorem. □

Proof of Theorem 7. We shall construct triangulations on non-orientable closed surfaces each of which quadrangulates a non-orientable closed surface. From Theorems 12 and 13 it is clear that the complete graph K_n with n vertices can triangulates and quadrangulates a non-orientable closed surfaces if and only if $n \equiv 0$, 1, 4 or $9 \bmod 12$. As in the proof of

Theorem 6, we have to investigate two cases. In the even case, we consider the two curves $y = h_3\left(\frac{1}{2}x\right) + \frac{3}{2} - 1$ and $y = h_2\left(\frac{1}{2}x\right) + \frac{3}{2} - 1$. In the odd case, we consider these too. $\square$

References

[1] N. Hartsfield, G. Ringel, Minimal quadrangulations of nonorientable surfaces, J. Combin. Theory Ser. A 50 (1989) 186–195.
[2] S. Lawrencenko, H. Yang, Determination of the 4-genus of a complete graph, Preprint.
[3] A. Nakamoto, S. Negami, K. Ota, Planar triangulations which quadrangulate other surfaces, European J. Combin. 25 (2004) 817–833.
[5] G. Ringel, Map Color Theorem, Springer, Berlin, 1974.
[7] Y. Suzuki, Triangulations on closed surfaces which quadrangulate other surfaces, submitted for publication.